HOMOLOGICAL DIMENSION OF CROSSED PRODUCTS *

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0 Introduction

Throughout this paper, k is a field, R is an algebra over k, and H is a Hopf algebra over k . We say that $R\#_\sigma H$ is the crossed product of R and H if $R\#_\sigma H$ becomes an algebra over k by multiplication:

$$(a\#h)(b\#g) = \sum_{h,g} a(h_1 \cdot b)\sigma(h_2, g_1)\#h_3 g_2$$

for any $a, b \in R, h, g \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$ (see, [2, Definition 7.1.1].)

Let $lpd({}_RM)$, $lid({}_RM)$ and $lfd({}_RM)$ denote the left projective dimension, left injective dimension and left flat dimension of left R-module M, respectively. Let $lgD(R)$ and $wD(R)$ denote the left global dimension and weak dimension of algebra R, respectively.

Crossed products are very important algebraic structures. The relation between homological dimensions of algebra R and crossed product $R\#_\sigma H$ is often studied. J.C.Mconnell and J.C.Robson in [4, Theorem 7.5.6] obtained that

$$rgD(R) = rgD(R * G)$$

for any finite group G with $\mid G \mid^{-1} \in k$, where $R * G$ is skew group ring. It is clear that every skew group ring $R * G$ is a crossed product $R\#_\sigma kG$ with trivial σ. Zhong

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Yi in [9] obtained that the global dimension of crossed product $R * G$ is finite when the global dimension of R is finite and some other conditions hold.

In this paper, we obtain that the global dimensions of R and the crossed product $R\#_\sigma H$ are the same; meantime, their weak dimensions are also the same, when H is finite-dimensional semisimple and cosemisimple Hopf algebra.

1 The homological dimensions of modules over crossed products

In this section, we give the relation between homological dimensions of modules over R and $R\#_\sigma H$.

If M is a left (right) $R\#_\sigma H$-module, then M is also a left (right) R-module since we can view R as a subalgebra of $R\#_\sigma H$.

Lemma 1.1 *Let R be a subalgebra of algebra A.*

(i) If M is a free A-module and A is a free R-module, then M is a free R-module;

(ii) If P is a projective left $R\#_\sigma H$-module, then P is a projective left R-module;

(iii) If P is a projective right $R\#_\sigma H$-module and H is a Hopf algebra with invertible antipode, then P is a projective right R -module;

(iv) If

$$\mathcal{P}_M \qquad : \cdots P_n \xrightarrow{d_n} P_{n-1} \cdots \to P_0 \xrightarrow{d_0} M \to 0$$

is a projective resolution of left $R\#_\sigma H$-module M, then $\mathcal{P}_M$ is a projective resolution of left R-module M;

(v) If

$$\mathcal{P}_M : \qquad \cdots P_n \xrightarrow{d_n} P_{n-1} \cdots \to P_0 \xrightarrow{d_0} M \to 0$$

is a projective resolution of right $R\#_\sigma H$-module M and H is a Hopf algebra with invertible antipode, then $\mathcal{P}_M$ is a projective resolution of right R-module M .

Proof. (i) It is obvious.

(ii) Since P is a projective $R\#_\sigma H$-module, we have that there exists a free $R\#_\sigma H$-module F such that P is a summand of F. It is clear that $R\#_\sigma H \cong R \otimes H$ as left R-module, which implies that $R\#_\sigma H$ is a free R-module. Thus it follows

from part (i) that F is a free R-module and P is a summand of F as R-module. Consequently, P is a projective R-module.

(iii) By [2, Corollary 7.2.11], $R\#_\sigma H \cong H \otimes R$ as right R-module. Thus $R\#_\sigma H$ is a free right R-module. Using the method in the proof of part (i), we have that P is a projective rigth R-module.

(iv) and (v) can be obtained by part (ii) and (iii). $\square$

Lemma 1.2 *(i) Let R be a subalgebra of A. If M is a flat right (left) A-module and A is a flat right (left) R-module, then M is a flat right (left) R-module;*

(ii) If F is a flat left $R\#_\sigma H$-module, then F is a flat left R-module;

(iii) If F is a flat right $R\#_\sigma H$-module and H is a Hopf algebra with invertible antipode, then F is a flat right R -module;

(iv) If

$$\mathcal{F}_M : \qquad \cdots F_n \xrightarrow{d_n} F_{n-1} \cdots \to F_0 \xrightarrow{d_0} M \to 0$$

is a flat resolution of left $R\#_\sigma H$-module M, then $\mathcal{F}_M$ is a flat resolution of left R-module M;

(v) If

$$\mathcal{F}_M : \qquad \cdots F_n \xrightarrow{d_n} F_{n-1} \cdots \to F_0 \xrightarrow{d_0} M \to 0$$

is a flat resolution of right $R\#_\sigma H$-module M and H is a Hopf algebra with invertible antipode, then $\mathcal{F}_M$ is a flat resolution of M;

Proof. (i) We only show part (i) in the case which M is a right A-module and A is a right R-module; the other cases can similarly be shown. Let

$$0 \to X \xrightarrow{f} Y$$

be an exact left $R\#_\sigma H$-module sequence. By assuptions,

$$0 \to A \otimes_R X \xrightarrow{A\otimes f} A \otimes_R Y$$

and

$$0 \to M \otimes_A (A \otimes_R X) \xrightarrow{M\otimes(A\otimes f)} M \otimes_A (A \otimes_R Y)$$

are exact sequences. Obviously,

$$M \otimes_A (A \otimes_R X) \cong M \otimes_R X \quad \text{ and } M \otimes_A (A \otimes_R Y) \cong M \otimes_R Y$$

as additive groups. Thus

$$0 \to M \otimes_R X \overset{M\otimes f}{\to} M \otimes_R Y$$

is an exact sequence, which implies M is a flat R-module.

(ii)-(v) are immediate consequence of part (i) $\Box$

The following is a immediate consequence of Lemma 1.1 and 1.2.

Proposition 1.3 *(i) If M is a left $R\#_\sigma H$-module, then*

$$lpd({}_RM) \leq lpd({}_{R\#_\sigma H}M);$$

(ii) If M is a right $R\#_\sigma H$-module and H is a Hopf algebra with invertible antipode, then

$$rpd(M_R) \leq rpd(M_{R\#_\sigma H})$$

(iii) If M is a left $R\#_\sigma H$-module, then

$$lfd({}_RM) \leq lfd({}_{R\#_\sigma H}M);$$

(iv) If M is a right $R\#_\sigma H$-module and H is a Hopf algebra with invertible antipode, then

$$rfd(M_R) \leq rfd(M_{R\#_\sigma H}).$$

Lemma 1.4 *Let H be a finite-dimensional semisimple Hopf algebra, and let M and N be left $R\#_\sigma H$-modules. If f is an R-module homomorphism from M to N, and*

$$\bar{f}(m) = \sum \gamma^{-1}(t_1) f(\gamma(t_2)m)$$

for any $m \in M$, then $\bar{f}$ is an $R\#_\sigma H$-module homomorphism from M to N, where $t \in \int_H^r$ with $\epsilon(t) = 1$, and γ is a map from H to $R\#_\sigma H$ sending h to $1\#h$.

Proof. (see, the proof of [2, Theorem 7.4.2]) For any $a \in R, h \in H, m \in M$, we see that

$$\begin{aligned}
\bar{f}(am) &= \sum \gamma^{-1}(t_1) f(\gamma(t_2)am) \\
&= \sum \gamma^{-1}(t_1) f((t_2 \cdot a)\gamma(t_3)m) \\
&= \sum \gamma^{-1}(t_1)(t_2 \cdot a) f(\gamma(t_3)m) \\
&= \sum a\gamma^{-1}(t_1) f(\gamma(t_2)m) \\
&= a\bar{f}(m)
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}(\gamma(h)m) &= \sum \gamma^{-1}(t_1) f(\gamma(t_2)\gamma(h)m) \\
&= \sum \gamma^{-1}(t_1) f(\sigma(t_2, h_1)\gamma(t_3 h_2)m) \quad \text{by [2, Definition 7.1.1]} \\
&= \sum \gamma^{-1}(t_1)\sigma(t_2, h_1) f(\gamma(t_3 h_2)m) \\
&= \sum \gamma(h_1)\gamma^{-1}(t_1 h_2) f(\gamma(t_2 h_3)m) \\
&= \sum \gamma(h)\gamma^{-1}(t_1) f(\gamma(t_2)m) \quad \text{ since } \sum h_1 \otimes t_1 h_2 \otimes t_2 h_3 = \sum h \otimes t_1 \otimes t_2 \\
&= \gamma(h)\bar{f}(m)
\end{aligned}$$

Thus $\bar{f}$ is an $R\#_\sigma H$-module homomorphism. $\Box$

In fact, we can obtain a functor by Lemma 1.4. Let ${}_{R\#_\sigma H}\overline{\mathcal{M}}$ denote the full subcategory of ${}_R\mathcal{M}$; its objects are all of left $R\#_\sigma H$-modules and its morphisms from M to N are all of R-module homomorphisms from M to N. For any $M, N \in ob_{R\#_\sigma H}\overline{\mathcal{M}}$ and R-module homomorphism f from M to N, we define that

$$F : {}_{R\#_\sigma H}\overline{\mathcal{M}} \longrightarrow {}_{R\#_\sigma H}\mathcal{M}$$

such that

$$F(M) = M \qquad \text{and} \qquad F(f) = \bar{f},$$

where $\bar{f}$ is defined in Lemma 1.4. It is clear that F is a functor.

Lemma 1.5 *Let H be a finite-dimensional semisimple Hopf algebra, and let M and N be right $R\#_\sigma H$-modules. If f is an R-module homomorphism from M to N, and*

$$\bar{f}(m) = \sum f(m\gamma^{-1}(t_1))\gamma(t_2)$$

for any $m \in M$, then $\bar{f}$ is an $R\#_\sigma H$-module homomorphism from M to N, where $t \in \int_H^r$ with $\epsilon(t) = 1$, γ is a map from H to $R\#_\sigma H$ sending h to $1\#h$.

Proof. (see, the proof of [2, Theorem 7.4.2]) For any $a \in R, h \in H, m \in M$, we see that

$$\begin{aligned}
\bar{f}(ma) &= \sum f(ma\gamma^{-1}(t_1))\gamma(t_2) \\
&= \sum f(m\gamma^{-1}(t_1)(t_2 \cdot a))\gamma(t_3) \\
&= \sum f(m\gamma^{-1}(t_1))(t_2 \cdot a)\gamma(t_3) \\
&= \sum f(m\gamma^{-1}(t_1))\gamma(t_2)a \\
&= \bar{f}(m)a
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}(m\gamma(h)) &= \sum f(m\gamma(h)\gamma^{-1}(t_1))\gamma(t_2) \\
&= \sum f(m\gamma(h_1)\gamma^{-1}(t_1h_2))\gamma(t_2h_3) \quad \text{since } \sum h_1 \otimes t_1h_2 \otimes t_2h_3 = \sum h \otimes t_1 \otimes t_2 \\
&= \sum f(m\gamma^{-1}(t_1)\sigma(t_2, h_1))\gamma(t_3h_2) \quad \text{by [2, Definition 7.1.1]} \\
&= \sum f(m\gamma^{-1}(t_1))\sigma(t_2, h_1)\gamma(t_3h_2) \\
&= \sum f(m\gamma^{-1}(t_1))\gamma(t_2)\gamma(h) \\
&= \bar{f}(m))\gamma(h)
\end{aligned}$$

Thus $\bar{f}$ is an $R\#_\sigma H$-module homomorphism. □

Proposition 1.6 *Let H be a finite-dimensional semisimple Hopf algebra.*

(i) If P is a left (right) $R\#_\sigma H$-modules and a projective left (right) R-module, then P is a projective left (right) $R\#_\sigma H$-module;

(ii) If E is a left (right) $R\#_\sigma H$-modules and an injective left (right) R-module, then E is an injective left (right) $R\#_\sigma H$-module;

(iii) If F is a left (right) $R\#_\sigma H$-modules and a flat left (right) R-module, then F is a flat left (right) $R\#_\sigma H$-module.

Proof. (i) Let

$$X \xrightarrow{f} Y \to 0$$

be an exact sequance of left (right) $R\#_\sigma H$-modules and g be a $R\#_\sigma H$-module homomorphism from P to Y. Since P is a projective left (right) R-module, we have that there exists a R-module homomorphism φ from P to X, such that

$$f\varphi = g.$$

By Lemma 1.4 and 1.5, there exists a $R\#_\sigma H$-module homomorphism $\bar{\varphi}$ from P to X such that

$$f\bar{\varphi} = g.$$

Thus P is a projective left (right) $R\#_\sigma H$-module.

Similarly, we can obtain the proof of part (ii).

(iii) Since F is a flat left (right) R-module, we have the character module $Hom_{\mathcal{Z}}(F, \mathcal{Q}/\mathcal{Z})$ of F is a injective left (right) R-module by [8, Theorem 2.3.6]. Obviously, $Hom_{\mathcal{Z}}(F, \mathcal{Q}/\mathcal{Z})$ is a left (right) $R\#_\sigma H$-module. By part (ii), $Hom_{\mathcal{Z}}(F, \mathcal{Q}/\mathcal{Z})$ is a injective left (right) $R\#_\sigma H$ -module. Thus F is a flat left (right) $R\#_\sigma H$-module. □

Proposition 1.7 *Let H be a finite-dimensional semisimple Hopf algebra. Then for left (right) $R\#_\sigma H$ -modules M and N,*

$$Ext^n_{R\#_\sigma H}(M, N) \subseteq Ext^n_R(M, N),$$

where n is any natural number.

Proof. We view the $Ext^n(M, N)$ as the equivalent classes of n- extension of M and N (see, [8, Definition 3.3.7]). We only prove this result for $n = 1$. For other cases, we can similarly prove. We denote the equivalent classes in $Ext^1_{R\#_\sigma H}(M, N)$ and $Ext^1_R(M, N)$ by [E] and [F]' , respectively, where E is an extension of $R\#_\sigma H$-modules M and N, and F is an extension of R-modules M and N. We define a map

$$\Psi : Ext^1_{R\#_\sigma H}(M, N) \to Ext^1_R(M, N), \quad \text{by sending } [E] \text{ to } [E]'.$$

Obviously, Ψ is a map. Now we show that Ψ is injective. Let

$$0 \to M \xrightarrow{f} E \xrightarrow{g} N \to 0 \quad \text{and} \quad 0 \to M \xrightarrow{f'} E' \xrightarrow{g'} M \to 0$$

are two extensions of $R\#_\sigma H$-modules M and N, and they are equivalent in $Ext^1_R(M, N)$. Thus there exists R-module homomorphism φ from E to E' such that

$$\varphi f = f' \quad \text{and} \quad \varphi g = g'.$$

By lemma 1.4 , there exists $R\#_\sigma H$-module homomorphism $\bar{\varphi}$ from E to E' such that

$$\bar{\varphi} f = f' \quad \text{and} \quad \bar{\varphi} g = g'.$$

Thus E and E' is equivalent in $Ext^1_{R\#_\sigma H}(M, N)$, which implies that Ψ is injective. □

Lemma 1.8 *For any $M \in \mathcal{M}_{R\#_\sigma H}$ and $N \in {}_{R\#_\sigma H}\mathcal{M}$, there exists an additive group homomorphism*

$$\xi : M \otimes_R N \to M \otimes_{R\#_\sigma H} N$$

by sending $(m \otimes n)$ to $m \otimes n$, where $m \in M, n \in N$.

Proof. It is trivial. $\Box$

Proposition 1.9 *If M is a right $R\#_\sigma H$-modules and N is a left $R\#_\sigma H$-module, then there exists additive group homomorphism*

$$\xi_* : \ Tor_n^R(M, N) \longrightarrow Tor_n^{R\#_\sigma H}(M, N)$$

such that $\xi_*([z_n]) = [\xi(z_n)]$, where ξ is the same as in Lemma 1.8.

Proof. Let

$$\mathcal{P}_M : \qquad \cdots P_n \stackrel{d_n}{\to} P_{n-1} \cdots \to P_0 \stackrel{d_0}{\to} M \to 0$$

is a projective resolution of right $R\#_\sigma H$-module M, and set

$$T = -\otimes_{R\#_\sigma H} N \quad \text{ and } \quad T^R = -\otimes_R N.$$

We have that

$$T\mathcal{P}_{\hat{M}} : \qquad \cdots T(P_n) \stackrel{Td_n}{\to} T(P_{n-1}) \cdots \to T(P_1) \stackrel{Td_1}{\to} TP_0 \to 0$$

and

$$T^R\mathcal{P}_{\hat{M}} : \qquad \cdots T^R(P_n) \stackrel{T^R d_n}{\to} T^R(P_{n-1}) \cdots \to T^R(P_1) \stackrel{T^R d_1}{\to} T^R(P_0) \to 0$$

are complexes . Thus ξ is a complex homomorphism from $T^R\mathcal{P}_{\hat{M}}$ to $T\mathcal{P}_{\hat{M}}$, which implies that ξ_* is an additive group homomorphism. $\Box$

2 The global dimensions and weak dimensions of crossed products

In this section we give the relation between homological dimensions of R and $R\#_\sigma H$.

Lemma 2.1 *If R and R' are Morita equivalent rings, then*
(i) rgD(R) = rgD(R');
(ii) lgD(R) = lgD(R');
(iii) wD(R)= wD(R').

Proof. It is an immediate consequence of [1, Proposition 21.6, Exercise 22.12] □

Theorem 2.2 *Let If H is a finite-dimensional semisimple Hopf algebra,*
(i) $rgD(R\#_\sigma H) \leq rgD(R)$;
(ii) $lgD(R\#_\sigma H) \leq lgD(R)$;
(iii) $wD(R\#_\sigma H) \leq wD(R)$.

Proof. (i) When $lgD(R)$ is infinite, obviously part (i) holds. Now we assume $lgD(R) = n$. For any left $R\#_\sigma H$-module M, and a projective resolution of left $R\#_\sigma H$-module M :

$$\mathcal{P}_M : \qquad \cdots P_n \xrightarrow{d_n} P_{n-1} \cdots \to P_0 \xrightarrow{d_0} M \to 0,$$

we have that $\mathcal{P}_M$ is also a projective resolution of left R-module M by Lemma 1.1. Let $K_n = ker\ d_n$ be syzygy n of $\mathcal{P}_M$. Since $lgD(R) = n$, $Ext_R^{n+1}(M, N) = 0$ for any left R-module N by [8, Corollary 3.3.6]. Thus $Ext^1_R(K_n, N) = 0$, which implies K_n is a projective R-module. By Lemma 1.6 (i), K_n is a projective $R\#_\sigma H$-module and $Ext^{n+1}_{R\#_\sigma H}(M, N) = 0$ for any $R\#_\sigma H$-module N. Consequently,

$$lgD(R\#_\sigma H) \leq n = lgD(R) \quad \text{by [8, Corollary 3.3.6]} .$$

We complete the proof of part (i).
We can similarly show part (ii) and part (iii). □

Theorem 2.3 *Let H be a finite-dimensional semisimple and cosemisimple Hopf algebra. Then*
(i) $rgD(R) = rgD(R\#_\sigma H)$;
(ii) $rgD(R) = rgD(R\#_\sigma H)$;
(iii) $wD(R) = wD(R\#_\sigma H)$.

Proof. (i) By dual theorem (see, [2, Corollary 9.4.17]), we have $(R\#_\sigma H)\#H^*$ and R are Morita equivanlent algebras. Thus $lgD(R) = lgD((R\#_\sigma H)\#H^*)$ by Lemma 2.1 (i). Considering Theorem 2.2 (i), we have that

$$lgD((R\#_\sigma H)\#H^*) \leq lgD(R\#_\sigma H) \leq lgD(R).$$

Consequently,

$$lgD(R) = lgD(R\#_\sigma H).$$

Similarly, we can prove (ii) and (iii) . □

Corollary 2.4 *Let H be a finite-dimensional semisimple Hopf algebra.*

(i) If R left (right) semi-hereditary, then so is $R\#_\sigma H$;

(ii) If R is von Neumann regular, then so is $R\#_\sigma H$.

Proof. (i) It follows from Theorem 2.2 and [8, Theorem 2.2.9] .

(ii) It follows from Theorem 2.2 and [8, Theorem 3.4.13]. □

By the way, part (ii) of Corollary 2.4 give one case about the semiprime question in [2, Question 7.4.9]. That is, If H is a finite-dimensional semisimple Hopf algebra and R is a von Neumann regular algebra (notice that every von Neumann regular algebra is semiprime), then $R\#_\sigma H$ is semiprime.

Corollary 2.5 *Let H be a finite-dimensional semisimple and cosemisimple Hopf algebra. Then*

(i) R is semisimple artinian iff $R\#_\sigma H$ is semisimple artinian;

(ii) R is left (right) semi-hereditary iff $R\#_\sigma H$ is left (right) semi-hereditary;

(iii) R is von Neumann regular iff $R\#_\sigma H$ is von Neumann regular.

Proof. (i) It follows from Theorem 2.3 and [8, Theorem 2.2.9].

(ii) It follows from Theorem 2.3 and [8, Theorem 2.2.9].

(iii) It follows from Theorem 2.3 and [8, Theorem 3.4.13]. □

If H is commutative or cocommutative, then $S^2 = id_H$ by [7]. Consequently, by [6, Proposition 2 (c)], H is semisimple and cosemisimple iff the character $chark$ of k does not divides $dimH$. Considering Theorem 2.3 and Corollary 2.5, we have:

Corollary 2.6 *Let H be a finite-dimensional commutative or cocommutative Hopf algebra. If the character chark of k does not divides $dimH$, then*

(i) $rgD(R) = rgD(R\#_\sigma H)$;

(ii) $rgD(R) = rgD(R\#_\sigma H)$;

(iii) $wD(R) = wD(R\#_\sigma H)$;

(iv) R is semisimple artinian iff $R\#_\sigma H$ is semisimple artinian;

(v) R is left (right) semi-hereditary iff $R\#_\sigma H$ is left (right) semi-hereditary;

(vi) R is von Neumann regular iff $R\#_\sigma H$ is von Neumann regular.

Since group algebra kG is a cocommutative Hopf algebra, we have that

$$rgD(R) = rgD(R * G).$$

Thus Corollary 2.6 implies in [4, Theorem 7.5.6].

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